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TUNNEL REINFORCEMENT VIA TOPOLOGY OPTIMIZATION

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SUMMARY

Anchoring is a fundamental method for supporting tunnels. It reinforces the original rock mass and reduces the deformation along the tunnel surface. The topological complexity of its layouts renders a design methodology difficult. A numerical approach to reinforce the host ground becomes desirable. The present paper proposes a topology optimization method based on a two-phase cell model and finite-element discretization of the host ground. The element consists of the original rock and the reinforcing material. The design issue involves the distribution of the reinforcing materials. The relative ratios of the two phases in various elements will be optimized to reduce the compliance of the tunnel.

The method enables the computer-aided design for the support of underground structures. The capabilities of the method are demonstrated by the designs to support a deep tunnel under various in situ stresses. The results indicate that oriented reinforcement is needed along the direction of the largest absolute value of the principal stress. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: tunnel reinforcement; topology optimization; two-phase cell; finite elements; compliance

1. INTRODUCTION

Supports for an excavation can be divided into two classes: passive support and active support, the latter emphasizes the use of ground itself. A fundamental form of active support is through rock bolts. The idea[1] that the host ground and support structures share the full load, and the host ground takes a major share of the load stimulates the emergence of the active support methodology. Lang and Bischoff[2] pioneered the designs of excavations that are based on the rock–bolt reinforcement of the rock mass. Bischoff and Smart[3] introduced a concept in which the use of rock–bolt reinforcement creates a uniform additional pressure on rock that is equivalent to that taken by steel ribs. The U.S. Army[4] discussed the design of rock bolts from a joint friction approach. Cording and Deere[5] introduced the concept of 'internal pressure' for the design of rock bolts to support a sliding critical wedge. A loosing zone theory based on the use of rock bolts is suggested by Dong *et al.*[6]

Nowadays designs of anchoring are still based on the past experiences or on the empirical recommendations. Fairhurst[7] pointed out that a partial reinforcement is preferred over a uniform one to stabilize the ground more efficiently. The ways of partial reinforcement are obviously

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case-specific. Active supports bring about the issue of reconstructing the ground around the tunnel. Anchoring can be regarded as the layouts of reinforcing materials to support the tunnel. The support design becomes the optimization of reinforcing materials in the design domain. The reinforcing materials, defined as the fully anchored rock mass, are stronger than the original rocks.

The layout of reinforcing materials involves topology optimization.[8,9] Topology optimization selects the best configuration for a structural system and is a rapidly expanding front of the structural design. The history of topology optimization can be traced back to as early as 1904 when Michell[10] developed the theory of the minimum weight truss design. Prager and Rozvany[11] provided various solutions and applications of the optimal layout theory. Based on the optimal configuration of a thin elastic plate with embedded microstructures by Cheng and Olhoff,[12] Bendsoe and Kikuchi[8] introduced the microstructures into the topology optimization. That advance broadened the practical applications of the structural optimization. The approach is known as the homogenization method for the topology optimization of a continuum structure.

The present work formulates the reinforcement on the host ground as a layout optimization to support a tunnel under the prescribed *in situ* stress. The finite-element method is employed and the ground compliance is minimized.[8,9] The problem is solved by a modified version of the scheme presented by Bendsoe and Kikuchi.[8]

2. MODELLING OF TUNNEL DESIGN

Consider a deep tunnel shown in Figure 1(a) subjected to the remote in-plane loading. The length of the tunnel validates the plane strain formulation. The original rock is regarded as the elastic material. In the absence of external supports, the stress analysis of the deep tunnel can be decomposed into two parts. One is the uniform stress field in Figure 1(b) prior to the introduction of the tunnel, and the other is the negating traction along the wall of the tunnel, as shown in Figure 1(c). The negating traction vector $\mathbf{t}$ relates to the *in situ* stress tensor $\boldsymbol{\sigma}$ by

$$\mathbf{t} = -\mathbf{n}\cdot\boldsymbol{\sigma} \tag{1}$$

where $\mathbf{n}$ is the outward normal to the tunnel wall. The negating traction $\mathbf{t}$ is not the traction on the support inserted later on. The latter combines the stresses at the rim of the support from two solutions of Figures 1(b) and 1(c).

For the in-plane case, the *in-situ* stresses can be expressed by

$$\begin{aligned} \sigma_x &= -k_0(1 + r\cos 2\alpha) \\ \sigma_y &= -k_0(1 - r\cos 2\alpha) \\ \sigma_{xy} &= -k_0 r \sin 2\alpha \end{aligned} \tag{2}$$

where k_0 is the hydrostatic pressure, r the ratio of the stress deviator to the hydrostatic pressure, α the angle-between x-axis and the largest absolute principal stress. When $0^\circ < \alpha < 45^\circ$, the horizontal stress is larger than the vertical one if $r > 0$, and *vice versa* if $r < 0$. The non-gravitational sources, such as tectonic stresses, residual stresses, or induced stresses (from man-made or geological structure), also contribute to the *in situ* stresses. Thus, the directions of the *in situ* stresses may vary in a wide range.

Consider the further deformation by the excavation of the tunnel. The reinforcement of the tunnel is aimed at reducing the extra deformation in Figure 1(c). The deformation field in

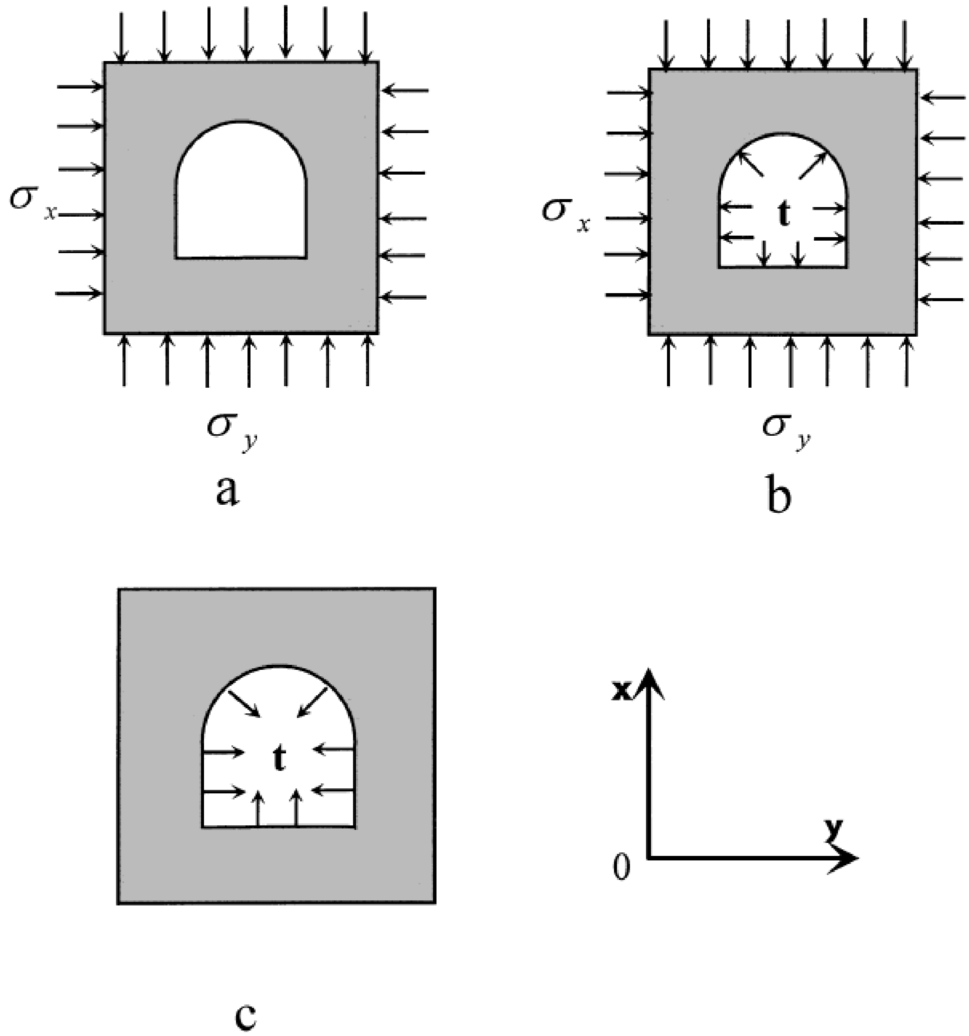

Figure 1. Stress analysis for a deep tunnel. (a) A tunnel subjected to remote in plane stresses; (b) uniform stress state before excavation; (c) the negating traction along the tunnel wall

Figure 1(c), due to the constraint by the surrounding original rocks, would decay exponentially from the tunnel. Therefore, we replace the infinite domain by a large but finite *design domain*. Its exterior boundary has zero normal displacement and is free of tangential constraint.

The deformation resisted by the supports is the displacement after excavation, plus the subsequent displacement by creep and/or by further loading due to construction activities. By considering the rock materials as elastic media, the deformation occurs prior to the installation of the support, raising doubt to the feasibility of the present approach. Careful examinations, however, provide some merit to the present approach. First, the reversibility of linearity elasticity makes no distinction for the support installed *a priori* or *a posteriori*. Second, for supports

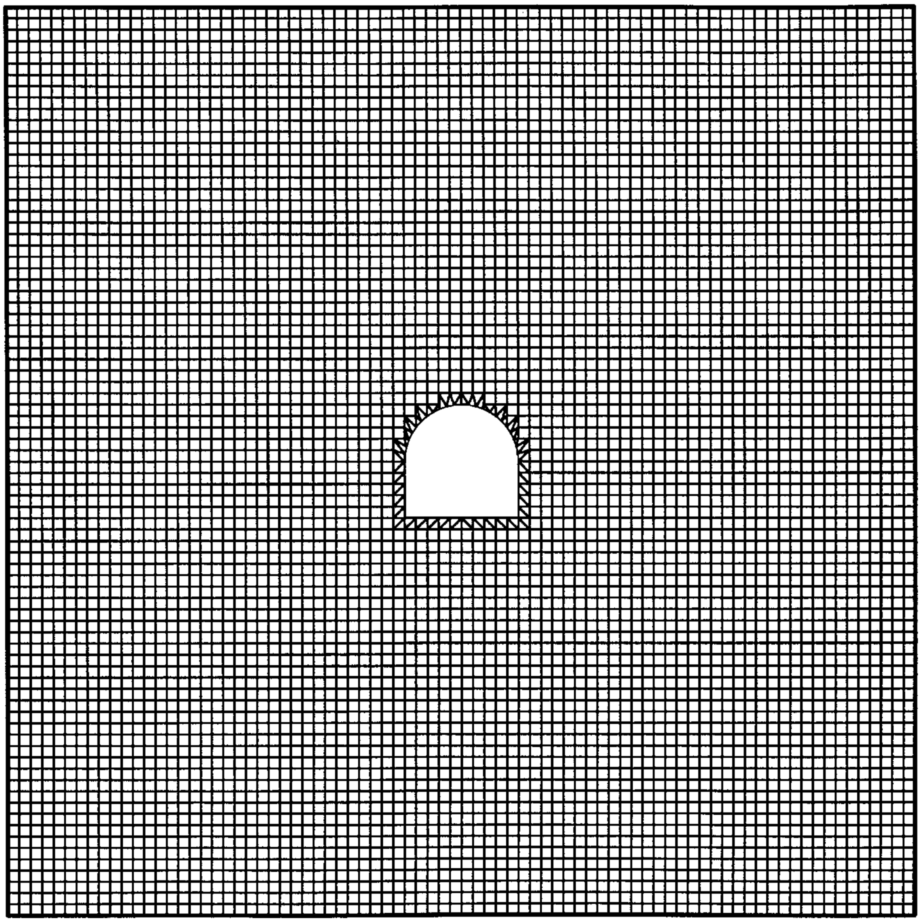

Figure 2. The design domain: the inner layer of the tunnel is replaced by the coating material, the square elements outside are the area to be designed

installed closed to a tunnel face, the present approach can handle the additional elastic loads (when the face is advanced and the 'dome-action' loads carried by the face are transferred onto the walls of excavation) quite well. Finally, time-dependent creep of the rock mass can lead to additional quasi-elastic loading of the support[13] which can be handled within the framework of the present methodology by defining a 'long-term' elastic modulus (see, for example, Muir Wood[14]). Previous discussions on elastic analysis for tunnel supports can be found in Fairhurst.[13]

To reinforce the host ground, we apply the topology optimization under a homogenization model. The design domain is discretized in Figure 2. The inner layer of the host ground is always replaced by a coating material, as indicated by the blackened elements in Figure 4. It can be referred as concrete or shotcrete. For simplicity, the coating material is selected to have the same elastic parameters as that of the reinforcing material. Optional reinforcement will be carried out

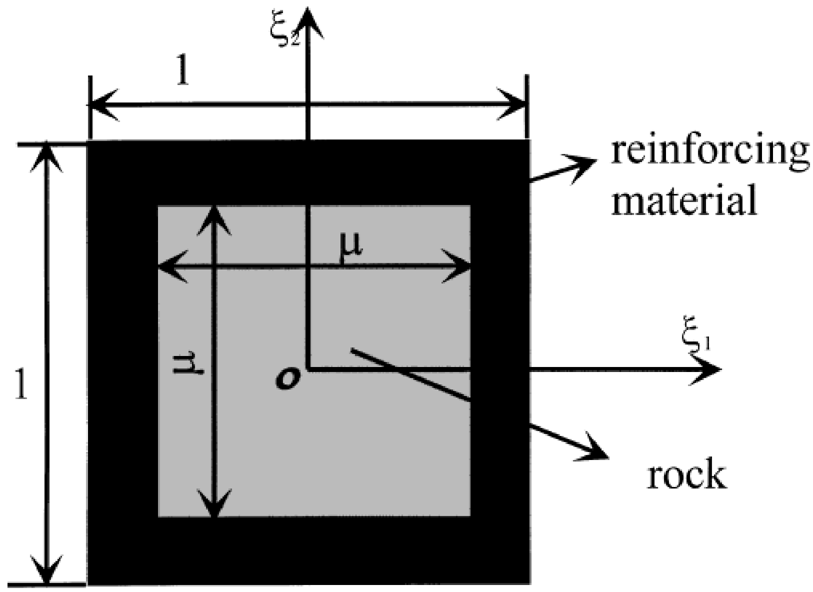

Figure 3. A square cell consisting of the original rock and the reinforcing material

in the rock domain outside this layer. In the design domain, every macroscopic point is represented by a composite cell of the reinforcing material and the original rock. We choose the square cell as shown in Figure 3, where the square inclusion is made of the original rock, surrounded by the reinforcing materials. The percentage of the reinforcing material, ρ, is given by

$$\rho = 1 - \mu^2 \tag{3}$$

where μ is the side-length of the inclusion in a unit cell. Either ρ or μ could change from 0 to 1, which means that the cell could be completely occupied by the original rock or by the reinforcing material. One could instead use a square cell where the reinforcing material is surrounded by the original rock. The two models are identical for the extreme cases of $\rho = 0$ and 1. The alternation only induces minor difference in all simulations reported in this paper. In most cases, the optimization leads to a sharp division of ρ values, either zero or one, with few elements in between.

For the best reinforcement of the design domain, one has to find the optimal distribution of ρ at all points within the domain. According to the homogenization method for topology optimization, the domain is discretized by finite elements. The problem is to determine the average percentage of the reinforcing material in each element such that the value of an objective function minimizes.

The corresponding mechanics analysis requires an expression for the constitutive tensor $E_{ijkl}(\mathbf{x})$, where $\mathbf{x}$ denotes the macroscopic location. The homogenization method evaluates the homogenized constitutive tensor $E_{ijkl}(\mathbf{x})$ by a perturbation analysis from a periodic field. The homogenized constitutive tensor is given by[8,9]

$$E_{ijkl}(\mathbf{x}) = \frac{1}{A}\int_A \left\{ E^{\xi}_{ijkl}(\mathbf{x}, \xi) - E^{\xi}_{ijpq}(\mathbf{x}, \xi)\frac{\partial \chi_{klp}}{\partial \xi_q} \right\} \mathrm{d}\xi \tag{4}$$

where ξ is the microscopic location within the cell which occupies an area A, as shown by the local co-ordinates ξ_1 and ξ_2 in Figure 3. The tensorial field $E^{\xi}_{ijkl}(\mathbf{x}, \xi)$ denotes the elasticity tensor at a microscopic point ξ within a cell macroscopically located at $\mathbf{x}$. According to the material occupied at point ξ, $E^{\xi}_{ijkl}(\mathbf{x}, \xi)$ at a microscopic scale will be either the elastic tensor of the reinforcing material or that of the original rock. If the analysis is targeted for the instantaneous response, the instantaneous elastic moduli should be used; on the other hand, the long-term 'remanent' moduli should be used if the long-term quasi-elastic creep response is of interest. The elastic tensor of the reinforcing material can be determined by a micromechanics model, which will be the task of further investigations. Simple estimates for the elastic properties of the reinforcing material can be derived by the classical Reuss (compliant) or Voigt (stiffness) assembly. The specific expressions for the elastic tensors of the reinforcing material and original rock will be prescribed in the example given in Section 4.

In equation (4), χ_{klp} denotes a microscopic displacement field under a unit macroscopic strain component ε_{kl}. It is evaluated by

$$\int_A \left[E^{\xi}_{ijpq}(\mathbf{x}, \xi) \frac{\partial \chi_{klp}}{\partial \xi_q} \right] \frac{\partial \phi_i}{\partial \xi_j} \, \mathrm{d}\xi = \int_A E^{\xi}_{ijkl}(\mathbf{x}, \xi) \frac{\partial \phi_i}{\partial \xi_j} \, \mathrm{d}\xi \quad \text{for all } \phi_i \in U_\xi \tag{5}$$

where ϕ_i denotes a kinematically admissible displacement field, and U_ξ the set of all virtual displacement fields satisfying the assembled cell periodicity. The readers may refer to the book by Bensoussan *et al.*[15] for a full description of the homogenization method.

3. OPTIMIZING THE GROUND REINFORCEMENT

The design for tunnel support endeavors a minimum amount of reinforcements while maintaining acceptable tunnel deformation. An alternating proposition would be the minimization of the tunnel displacement under the prescribed volume of the reinforcing material. A statement for the latter is "*an objective function that measures the deformation of the tunnel surface should be minimized under a set of constraints*'.

We now make this statement precise. Since the negating traction vector $\mathbf{t}$ is constant along the tunnel wall, the external work along the tunnel wall Γ

$$W(\mathbf{u}) = \int_\Gamma \mathbf{t} \cdot \mathbf{u} \, \mathrm{d}\Gamma \tag{6}$$

can be chosen as an appropriate *objective function* that measures the deformation of the tunnel wall. The work by the external force has been widely used as the objective function in topology optimization and the existence of solution is well established. The objective function W is usually termed the 'compliance' of the structure.

The *set of constraints* can be listed as follows: (a) the equilibrium equation within the structure, (b) the constraint on the total volume of reinforcing materials, and (c) a bound on the design variable, $0 \leqslant \mu \leqslant 1$. The equilibrium equation of the structure is usually cast in its weak form:

$$a(\mathbf{u}, \mathbf{v}) = W(\mathbf{v}) \quad \forall \mathbf{v} \in U, \quad \mathbf{E} \in E_{\mathrm{ad}} \tag{7}$$

where $a(\mathbf{u}, \mathbf{v}) = \int_\Omega E_{ijkl} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{kl}(\mathbf{v}) \, \mathrm{d}\Omega$, with ε_{ij} being the strain tensor given by the symmetric part of the displacement gradient. The auxiliary variable $\mathbf{v}$ denotes a kinematically admissible

displacement field. The constitutive tensor E_{ijkl} is given by (4) and belongs to an admissible set E_{ad}. The symbol U denotes the space of kinematically admissible displacements and Ω the design domain. The volume constraint on the reinforcing materials can be phrased as

$$\int_{\Omega} (1-\mu^2)\,\mathrm{d}\Omega \leqslant V \tag{8}$$

where V is the prescribed total volume of reinforcing materials.

The optimization problem can be summarized as

$$\min_{\mathbf{u}\in U,\ \mathbf{E}\in E_{\mathrm{ad}}} W(\mathbf{u})$$

subject to:

$$a(\mathbf{u},\mathbf{v}) = W(\mathbf{v}) \quad \forall \mathbf{v}\in U, \quad \mathbf{E}\in E_{\mathrm{ad}}, \quad \int_{\Omega}(1-\mu^2)\,\mathrm{d}\Omega \leqslant V \quad 0\leqslant \mu \leqslant 1$$

The necessary conditions for the optimality of the sizing variable μ requires that the following Lagrange function be stationary:

$$L = W(\mathbf{u}) - [a(\mathbf{u},\mathbf{v}) - W(\mathbf{v})] + \Lambda\left[\int (1-\mu^2)\,\mathrm{d}\Omega - V\right] + \int z^{+}(\mu-1)\,\mathrm{d}\Omega - \int z^{-}\mu\,\mathrm{d}\Omega \tag{9}$$

where the kinematically admissible field $\mathbf{v}$ is regarded as the Lagrangian multiplier for equilibrium equations, z^- and z^+ the Lagrangian multipliers for the lower and upper bounds of the design variable, Λ the Lagrangian multiplier for the total volume constraint. The necessary condition for optimality gives $\mathbf{u}=\mathbf{v}$. In the region of intermediate μ values, $0<\mu<1$, a supplementary condition for μ should be observed

$$-\frac{1}{2\Lambda\mu}\frac{\partial E_{ijkl}}{\partial \mu}\,\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{u}) = 1 \tag{10}$$

Equation (10) may offer numerous iteration routines. After trials and errors for a converging scheme, we modify (10) to the following iteration routine:

$$\left[-\frac{1}{2\Lambda^{(m)}\mu^{(m)}}\frac{\partial E^{(m)}_{ijkl}}{\partial \mu}\,\varepsilon_{ij}(\mathbf{u}^{(m)})\varepsilon_{kl}(\mathbf{u}^{(m)})\right]^q = \frac{1-\mu^{(m+1)}}{1-\mu^{(m)}} \tag{11}$$

where q is a tuning parameter and the superscript (m) marks the number of iteration. The influence of q was discussed by Bendsoe.[8,9] The Lagrange multiplier $\Lambda^{(m)}$ will be derived later in (15). The factor $(1-\mu^{(m+1)})/(1-\mu^{(m)})$ will converge to unity and therefore bear no effect on the final result. Equation (11) enables an updating scheme for the design variable μ:

$$\mu^{(m+1)} = \begin{cases} 0 & \text{if} \quad A^{(m)} \leqslant \max\{(1-\varsigma)\mu^{(m)}, 0\} \\ A^{(m)} & \text{if} \quad \max\{(1-\varsigma)\mu^{(m)},0\} \leqslant A^{(m)} \leqslant \min\{(1+\varsigma)\mu^{(m)}, 1\} \\ 1 & \text{if} \quad \min\{(1+\varsigma)\mu^{(m)},1\} \leqslant A^{(m)} \end{cases} \tag{12}$$

where ς is a move limit, see References 8 and 9 for further details. The top and the lower choices in the scheme (12) come from the lower and the upper bounds of μ at zero and one. The values of q and ς are chosen for rapid and stable convergence of the scheme. In most numerical simulations to follow, we choose $q = 0{\cdot}8$ and $\varsigma = 0{\cdot}5$. The expression of $A^{(m)}$ is given by

$$A^{(m)} = 1 - (1 - \mu^{(m)}) \left(\frac{B^{(m)}}{\Lambda^{(m)} \mu^{(m)}} \right)^q \tag{13}$$

with

$$B^{(m)} = -\frac{1}{2} \frac{\partial E_{ijkl}^{(m)}}{\partial \mu} \varepsilon_{ij}(\mathbf{u}^{(m)}) \varepsilon_{kl}(\mathbf{u}^{(m)}) \tag{14}$$

The Langrange multiplier Λ can be obtained from (8) as

$$\frac{1}{\Lambda^{(m)^q}} = \frac{D^{(m)} - \sqrt{D^{(m)^2} - V_{\text{act}} \left[\sum_{e \in R_{\text{act}}} (1 - \mu^{(m)})^2 \left(\frac{B^{(m)}}{\mu^{(m)}} \right)^{2q} \right]}}{\sum_{e \in R_{\text{act}}} (1 - \mu^{(m)})^2 \left(\frac{B^{(m)}}{\mu^{(m)}} \right)^{2q}} \tag{15}$$

with

$$D^{(m)} = \sum_{e \in R_{\text{act}}} (1 - \mu^{(m)}) \left(\frac{B^{(m)}}{\mu^{(m)}} \right)^q \tag{16}$$

where the element number e is summed over a subset R_{act} with intermediate μ values and V_{act} denotes the volume of the reinforcing materials in subset R_{act}.

Because the subset R_{act} is unknown, an inner iteration loop composed of equations (12) and (15) should be executed. The close-form expression for Λ renders the scheme easy to implement. The numerical scheme can be summarized as follows:

1. *Initial data assignment*: The initial design for the host ground reinforcements is defined as follows: the tunnel rim (one layer of triangular elements) is replaced by the coating material and the prescribed total volume of reinforcing materials is distributed uniformly in the square elements shown by Figure 2. That is, all square elements in Figure 2 have the same initial percentage ρ_0.
2. *Boundary value problem*: The exterior boundary of the design domain is free of tangential constraint but confined rigidly in the normal direction. The tunnel surface is subjected to the negating traction (1) with stress tensor $\boldsymbol{\sigma}$ given in (2).
3. *Cell calculation*: A micromechanics calculation is conducted to establish a relationship between the elastic tensor $\mathbf{E}$ and the design variable μ by (4) and (5).
4. *Prediction*: The finite-element equation is solved and the convergence checked under the following criterion:

$$\frac{|W^{(m+1)} - W^{(m)}|}{W^{(m+1)}} \leqslant \delta \tag{17}$$

   where $W^{(m)}$ and $W^{(m+1)}$ are the previous and the current values of the external work, and δ is a given tolerance. A tolerance value of $\delta = 10^{-4}$ will be used in the subsequent calculations.
5. *Correction*: The design variables are updated by equation (12), if not converged.

4. RESULTS AND DISCUSSIONS

This section will explore the designs of tunnel supports under different *in situ* stresses. In all examples to follow, the total volume of reinforcing materials is designated as 5 per cent of the design domain, namely $\rho_0 = 5$ per cent. Isotropic material response will be used. The reinforcing

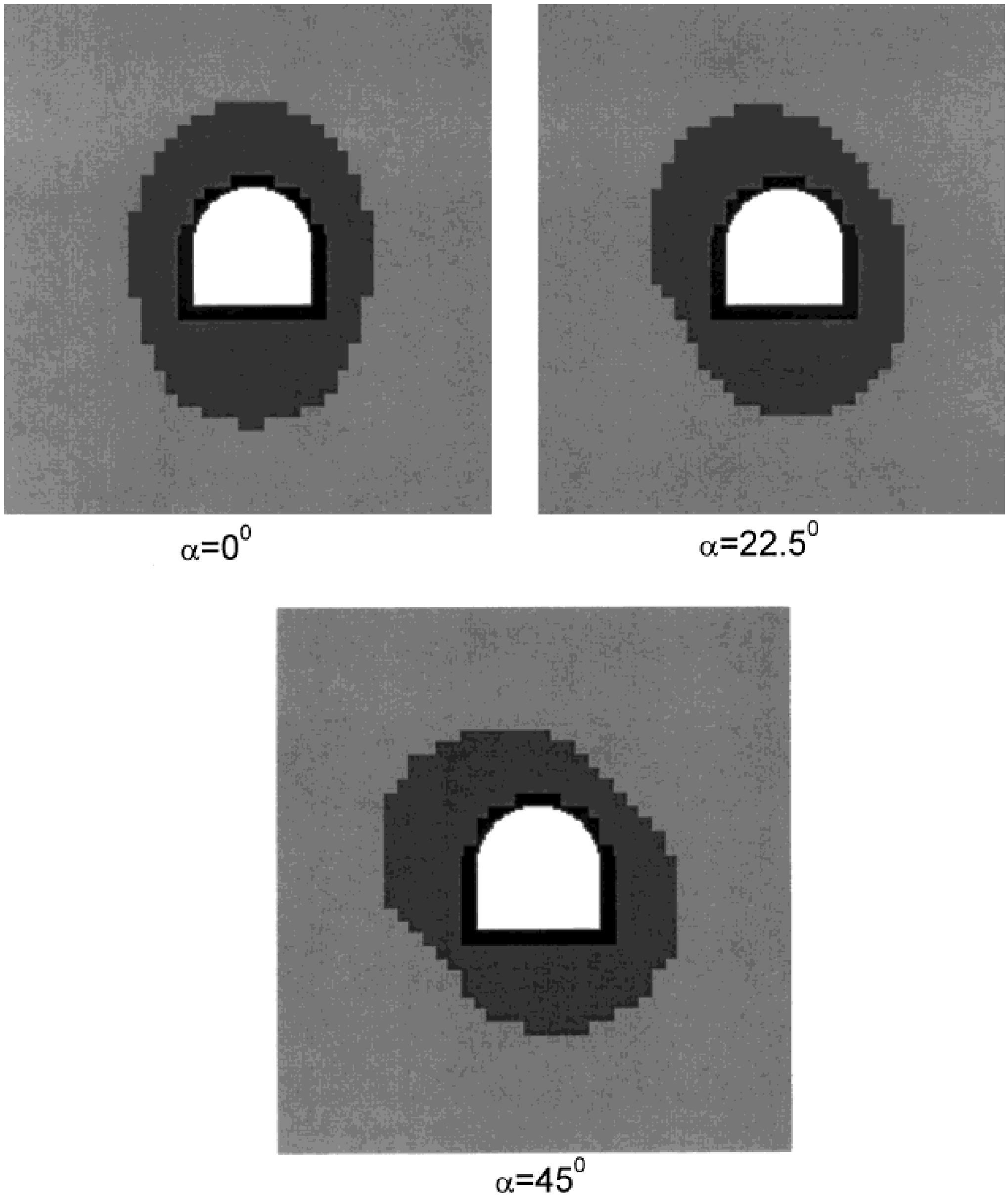

Figure 4. Optimal distribution of reinforcing materials under the *in situ* stress states of $r = -0{\cdot}35$, $\alpha = 0°$, $22{\cdot}5°$ and $45°$

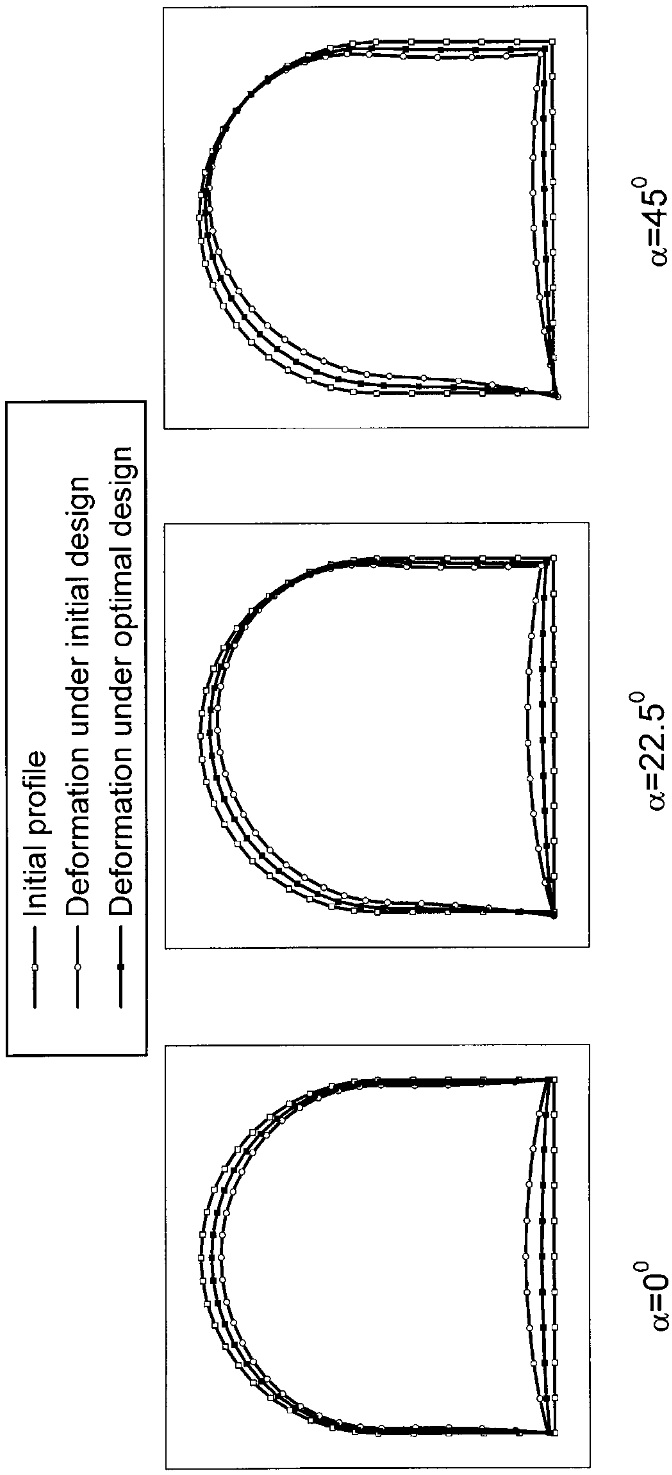

Figure 5. Deformations of the tunnel profiles under the initial and the optimal designs subjected to the *in situ* stress states of $r = -0{\cdot}35$, $\alpha = 0°$, $22{\cdot}5°$ and $45°$

material possesses a Young's modulus five times of that of the original rock. The Poisson's ratio is 0·25 for the original rock and 0·3 for the reinforcing material. The tunnel is located deep enough so that the difference of the gravity force is negligible. The rock mass around the tunnel is assumed to be homogeneous. Accordingly, the distribution of reinforcing materials only depends on the parameters of r and α in equation (2). Hoek and Brown[16] reported that the ratio σ_x/σ_y ranges from 0·48 to 5·56, corresponding to a variation of r from $-0{\cdot}35$ to 0·7. Various values of r are explored under three angles $\alpha = 0$, 22·5 and 45°. Here, we only present the case of $r = -0{\cdot}35$ under the above-mentioned three angles. The common features of the optimal distribution of reinforcing material can be inferred from this case.

Figure 4 depicts the optimal reinforcement of the host ground. Black colour denotes the *a priori* prescribed area of the coating materials, light grey colour denotes the original rock material ($\rho = 0$), and dark grey colour denotes the reinforcing material ($\rho = 1$). The value of ρ is found to be either 0 or 1 in most elements. One observes that the reinforcements are aligned with the principal stress of the largest absolute value. Deformations of the tunnel under the initial design and the optimal design are compared in Figure 5. The deformation of the tunnel under the optimal design is reduced approximately by 50 per cent from the one under the initial design. The declination of the 'compliance' is shown in Figure 6. A quick and uniform convergence is

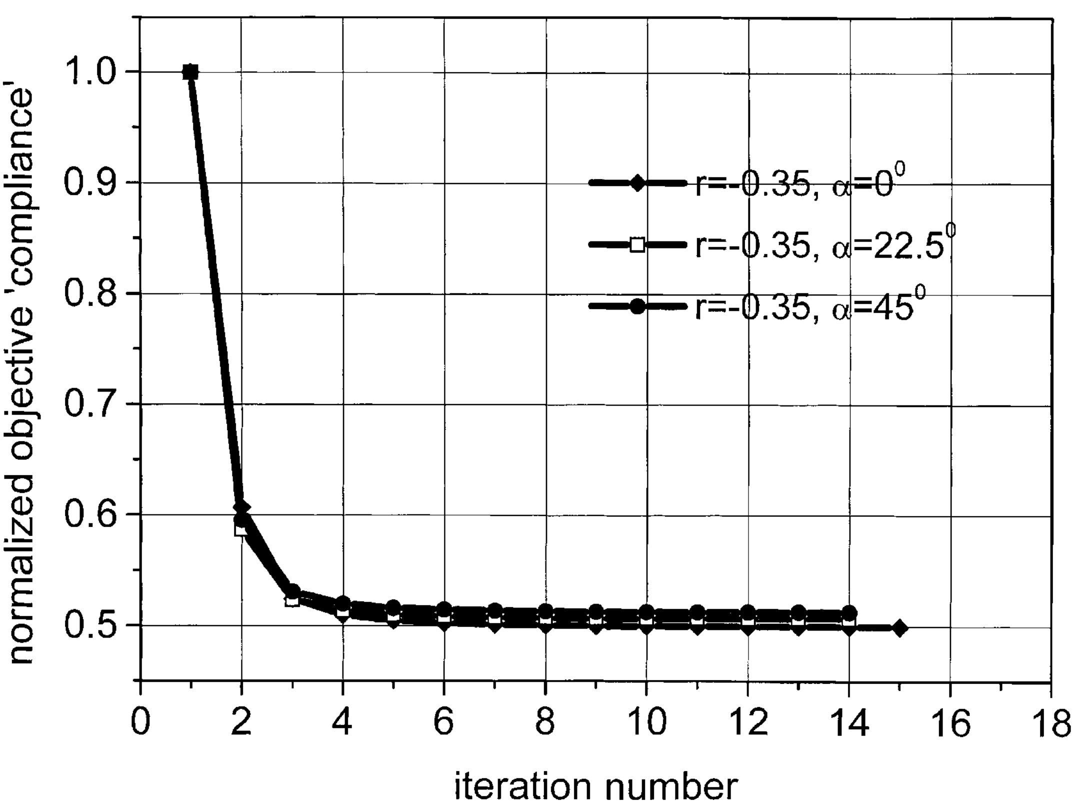

Figure 6. Convergence of compliances in various calculations

observed, and convergence is achieved within 15 iteration steps. The main improvement is obtained within 4 iterations and the results are almost stabilized after eight iterations.

5. CONCLUDING REMARKS

In this paper, we approach the design of tunnel supports by reinforcing the host ground under topology optimization. This approach makes the computer-aided design of tunnel supports feasible.

Numerical demonstrations for the present scheme are focused on the influence of *in situ* stresses, with the selected examples exploring the parameter α. Combined with other numerical results which are not reported here due to the lack of space, we arrive at the following conclusions. When the tunnel is subjected to a hydrostatic stress state, the optimal reinforcements are evenly distributed around the tunnel. Under a deviatoric stress state, on the other hand, the optimal reinforcements are aligned with the principal stress of the maximum absolute value. In the cases examined, the deformations of the tunnel under optimal designs are roughly reduced by 50 per cent from the deformations of the same tunnel under the initial design.

The generalization of the present work to three-dimensional configurations is straightforward. Furthermore, by the element-wise assignment of the elastic tensor to the original rock, the heterogeneity of the original rock mass can be accommodated in the present framework. The present scheme to determine the μ values element-wise would be consistent to the variable elasticity tensor scheme.

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